Inequalities Relating Degrees of Adjacent Vertices to the Average Degree

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The main theorem gives a class of inequalities concerning finite hypergraphs with a fixed number of vertices per edge. One corollary of the general result: if there are edges, then there is an edge such that the geometric mean of the degrees of its vertices is greater than or equal to the arithmetic mean degree of all the vertices. If the hypergraph is not regular, the inequality is strict.

Another corollary of the main result is used to derive an inequality for simple graphs relating the number of edges, the number of vertices, and the largest number of triangles based on a fixed edge in the graph. Finally, some results are derived relating the degrees of non-adjacent vertices of a simple graph to the average degree.

The objects of study in this paper are hypergraphs with a fixed number of vertices per edge. We wish to allow 'multiple edges', and to allow vertices to occur more than once in an edge. Of the various ways of defining what such a hypergraph is, the following suits our purpose best.

DEFINITION. A hypergraph with n vertices per edge (n-hypergraph, for short) is a matrix of non-negative integers with constant row sum n.

Given such a matrix $\mathbf{A}=[a_{ij}]$, $q\times p$, think of the columns as representing the vertices and the rows as representing the edges of the hypergraph; a_{ij} is the number of times the jth vertex occurs on the ith edge. The requirement that $\sum_{j=1}^{p} a_{ij}=n$, $i=1,\dots,q$, says that each edge is composed of n vertices, counting repetitions. The degree or valency of a vertex v, denoted $d(v)$, is the sum down the column of $\mathbf{A}$ corresponding to v. (We could take the view that the columns of $\mathbf{A}$ *are* the vertices, but we will tend to say that they represent, or correspond to, the vertices.) Note that, if the vertices are $v_1,\dots,v_p$, then

$$nq=\sum_{i=1}^{p} d(v_i).$$

If the column sums of $\mathbf{A}$ (the degrees of the vertices) are all the same, then $\mathbf{A}$ is said to be regular. If a column sum is zero, the corresponding vertex is said to be isolated. The average degree of the vertices of $\mathbf{A}$ will be denoted $\bar{d}(\mathbf{A})$, or just $\bar{d}$, if $\mathbf{A}$ is fixed in the discussion. Thus

$$\bar{d}=\frac{1}{p}\sum_{i=1}^{p} d(v_i)=\frac{nq}{p}.$$

It is good to keep in mind that a 2-hypergraph is an ordinary graph, with, possibly, loops and multiple edges. Also note that our formulation leaves the n-hypergraphs with no edges unaccounted for. The loss is not great.

The proof of our main result will use Jensen's inequality (see [2], pp. 70–74) in the following form: if f is a convex function on a real interval I, and $x_1\leqslant\cdots\leqslant x_m$ are points of I, then

$$f\left(\frac{1}{m}\sum_{i=1}^{m} x_i\right)\leqslant\frac{1}{m}\sum_{i=1}^{m} f(x_i),$$

0195-6698/86/030237+05 \$02.00/0

with equality only if f is linear (first degree polynomial) on $[x_1, x_m]$. Consequently, if equality holds, and f is not linear on any subinterval of I, then $x_1 = x_m = x_i$, $i = 1, \ldots, m$.

THEOREM. *Suppose $\mathbf{A} = [a_{ij}]$ is a $q \times p$ matrix of real numbers with constant row sum r, and column sums $d_1, \ldots, d_p$. Let $\bar{d} = (1/p)\sum_{i=1}^{p} d_i (= qr/p)$. Suppose ϕ is a function such that $f(x) = x\phi(x)$ is convex on an interval I containing $d_1, \ldots, d_p$. Then*

$$\begin{aligned}\max_i \sum_{j=1}^{p} a_{ij}\phi(d_j) &\geq \frac{1}{q}\sum_{i=1}^{q}\sum_{j=1}^{p} a_{ij}\phi(d_j)\\ &\geq r\phi(\bar{d}).\end{aligned}$$

Furthermore, if f is not linear on any subinterval of I, then the second inequality above is strict unless the column sums $d_1, \ldots, d_p$ are equal.

PROOF. The first inequality is an instance of the elderly truth that a maximum of numbers of greater than or equal to their arithmetic mean. For the second inequality, observe that

$$\begin{aligned}\frac{1}{q}\sum_{i=1}^{q}\sum_{j=1}^{p} a_{ij}\phi(d_j) &= \frac{1}{q}\sum_{j=1}^{p}\left(\sum_{i=1}^{q} a_{ij}\right)\phi(d_j)\\ &= \frac{1}{q}\sum_{j=1}^{p} d_j\phi(d_j)\\ &= \frac{1}{q}\sum_{j=1}^{p} f(d_j)\\ &\geq \frac{p}{q} f\left(\frac{1}{p}\sum_{j=1}^{p} d_j\right) \quad \text{[by Jensen's Inequality]}\\ &= \frac{p}{q} f(\bar{d}) = \frac{p}{q}\bar{d}\phi(\bar{d})\\ &= r\phi(\bar{d}).\end{aligned}$$

If f is not linear on any subinterval of I, then equality would force the d_j to be equal, by the equality condition in Jensen's Inequality.

COROLLARY 1. *Suppose $\mathbf{A} = [a_{ij}]$, $q \times p$, is an n-hypergraph $(n \geq 1)$ with column sums $d_1, \ldots, d_p$. Then, provided we agree that $0^0 = 1$,*

$$\max_i \left(\prod_{j=1}^{p} d_j^{a_{ij}}\right)^{1/n} \geq \left(\prod_{i=1}^{q}\left(\prod_{j=1}^{p} d_j^{a_{ij}}\right)^{1/n}\right)^{1/q} \geq \bar{d}.$$

Furthermore, the second inequality is strict if $\mathbf{A}$ is not regular.

REMARK. This result is a good deal more memorable if it is read in the language of vertices and edges. Keep in mind that each edge involves n vertices, not necessarily distinct; given the 'meaning' of a_{ij}, for each i, $(\prod_{j=1}^{p} d_j^{a_{ij}})^{1/n}$ is the geometric mean of degrees of the vertices on the ith edge, counting each vertex separately as many times as it appears on that edge. Thus, $[\prod_{i=1}^{q} (\prod_{j=1}^{p} d_j^{a_{ij}})^{1/n}]^{1/q}$ is the geometric mean, over the edges, of the geometric means of the degrees of the vertices involved in those edges, respectively. The main claim of the corollary is that this geometric mean is strictly greater than the arithmetic mean degree of the vertices of the hypergraph, when the hypergraph is not regular.

PROOF. First assume that $\mathbf{A}$ has no isolated vertices (zero columns), so that $d_j > 0$, $j = 1, \dots, p$. Apply the theorem with $r = n$ and $\phi(x) = \ln x$. We leave it to the reader to check that $x \ln x$ is strictly convex on $(0, \infty)$, and to write down the result of the theorem, divide through by n, and exponentiate, to arrive at the desired conclusion.

If $\mathbf{A}$ has isolated vertices, form a new matrix $\tilde{\mathbf{A}}$ by deleting the zero columns of $\mathbf{A}$; $\tilde{\mathbf{A}}$ is still an n-hypergraph, and the geometric means of the degrees of the vertices on its edges are the same as those of $\mathbf{A}$. (The convention that $0^0 = 1$ enters here.) Clearly $\bar{d}(\tilde{\mathbf{A}}) > \bar{d}(\mathbf{A})$, which implies the desired conclusion.

REMARK. For n-hypergraphs without isolated vertices, a more potent but less memorable inequality is obtained by taking $\phi(x) = \ln(1 + \ln x)$.

COROLLARY 2. *Suppose* $\mathbf{A} = [a_{ij}]$, $q \times p$, *is an* n*-hypergraph* $(n \geq 1)$ *with column sums* $d_1, \dots, d_p$. *Suppose* s *is a real number. Then*

$$\begin{aligned} \max_i \frac{1}{n} \sum_{j=1}^{p} a_{ij} d_j^s &\geq \frac{1}{q} \sum_{i=1}^{q} \left(\frac{1}{n} \sum_{j=1}^{p} a_{ij} d_j^s \right) \\ &\geq \bar{d}^s, \end{aligned}$$

provided either

(a) $s > 0$, *or*

(b) $s \leq -1$ *and* $\mathbf{A}$ *has no zero columns* (*isolated vertices*). *Furthermore, except in the case* $s = -1$, *in* (b), *we have equality throughout only if* $\mathbf{A}$ *is regular.*

PROOF. *Take* $\phi(x) = x^s$, *in the theorem.*

REMARKS. The reader is urged to restate this result in terms of edges, vertices, and degrees. Toward an understanding of the result in this form, it might be useful to check that the path of length 2, P_2, provides an example that shows that the second inequality of Corollary 2 need not hold, for each $s \in (-1, 0)$, and that for $s = -1$ we may have equality without regularity. (In the case of P_2 we have $n = 2$ and $\bar{d} = \frac{4}{3}$. In matrix form, $\mathbf{P}_2 = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right)$, not to be confused with an adjacency matrix of P_2.)

Our last three results are about simple graphs, and we shall revert to the practice of considering such a graph to be a pair (V, E), V the set of vertices, E the set of edges, a set of unordered pairs of V. Throughout, $p = |V|$ and $q = |E|$.

COROLLARY 3. *Suppose that* $G = (V, E)$ *is a finite simple graph with the property that each edge is a side of no more than* t *triangles. Then* $q \leq (p(p+t)/4)$.

Furthermore, equality implies that G *is regular, and that each edge is a side of* t *triangles.*

PROOF. We may as well suppose that $E \neq \emptyset$. We can then invoke Corollary 2, with $n = 2$, $s = 1$, and conclude that there are adjacent vertices u, v, such that

$$\bar{d} = \frac{2q}{p} \leq \frac{d(u) + d(v)}{2}.$$

Furthermore, if G is not regular, we can get strict inequality at this point.

Since u and v are adjacent, and G is simple, by the hypothesis there can be no more than t vertices to which u and v are both adjacent, so $d(u) + d(v) \leq p + t$. This completes the proof of the inequality, and the assertion that equality implies regularity.

Suppose equality holds and e is an edge with vertices u, v at its ends. We know that G is regular, so $d(u)=d(v)=\bar{d}=2q/p$. Suppose e is a side of t' triangles, $t'\leqslant t$. Then

$$\begin{aligned} p+t=\frac{4q}{p}&=d(u)+d(v)\\ &\leqslant p+t'\leqslant p+t. \end{aligned}$$

Thus $t=t'$. This completes the proof.

The case $t=0$ in Corollary 3 is a well-known theorem proven in 1907 by Mantel and others (see [1], Chapter 6). In that case, it is easy to see that equality holds in the conclusion only if G is the complete bipartite graph $K_{m,m}$, for some m. When t has its largest possible reasonable value, $p-2$, equality holds in Corollary 3 only if G is the complete graph on p vertices. It would be interesting to have descriptions of the graphs for which equality holds, in Corollary 3, for other values of t. (Our thanks to Anthony Hilton for his comments on this point, as well as for numerous other stimulating remarks.)

If, in Corollary 3, one takes t to be the number of triangles in G, the inequality obtained is usually much worse than those to be found in [1], Chapter 6, and is never better. However, it is clear that the smallest t for a given G which will satisfy the hypothesis of Corollary 3 is usually much smaller than the total number of triangles in G.

The last two corollaries relate the degrees of non-adjacent vertices of a finite simple graph $G=(V,E)$ to $\bar{d}=\bar{d}(G)$. There is an endless supply of such results obtainable by applying the Theorem to the complement $\tilde{G}=(V,\tilde{E})$ of G. (Two distinct vertices are adjacent in $\tilde{G}$ if and only if they are not adjacent in G.) The two results to be given here seem to us to be the best and most interesting of those found so far.

Throughout, let $\tilde{q}=|\tilde{E}|=(p(p-1)/2)-q$. For $u\in V$, $d(u)$ will still denote the degree of u in G; the degree in $\tilde{G}$ is $d_{\tilde{G}}(u)=p-1-d(u)$. Likewise, $\bar{d}(\tilde{G})=p-1-\bar{d}$. For any edge e, let $u(e)$, $v(e)$ denote the vertices on the edge.

COROLLARY 4. *Suppose $G=(V,E)$ is a finite simple graph, not complete. Then*

$$\begin{aligned} \min_{e\in\tilde{E}}\frac{d(u(e))+d(v(e))}{2}&\leqslant\frac{1}{\tilde{q}}\sum_{e\in\tilde{E}}\frac{d(u(e))+d(v(e))}{2}\\ &\leqslant\bar{d}. \end{aligned}$$

Furthermore, if G is not regular, the second inequality is strict.

PROOF. Only the second inequality needs proof. By Corollary 2, applied to $\tilde{G}$, with $n=2$, $s=1$,

$$\frac{1}{\tilde{q}}\sum_{e\in\tilde{E}}\frac{d_{\tilde{G}}(u(e))+d_{\tilde{G}}(v(e))}{2}\geqslant\bar{d}(\tilde{G}),$$

and if G, and thus $\tilde{G}$, is not regular, the inequality is strict. The proof is complete by plugging $p-1-d(u)$ for $d_{\tilde{G}}(u)$, for each vertex u, and $p-1-\bar{d}$ for $\bar{d}(\tilde{G})$, and rearranging the inequality.

COROLLARY 5. *Suppose $G=(V,E)$ is a finite simple graph, neither edgeless nor complete, and ϕ is a function on an interval I containing the degrees of the vertices of G such that*

(a) *ϕ is concave on I and*

(b) *$f(x)=x\phi(x)$ is convex on I.*

Then

$$\min_{e \in \tilde{E}} [\phi(d(u(e))) + \phi(d(v(e)))]$$
$$\leq \frac{1}{\tilde{q}} \sum_{e \in \tilde{E}} [\phi(d(u(e))) + \phi(d(v(e)))]$$
$$\leq 2\phi(\bar{d}).$$

Furthermore, if either ϕ or f has the property of being non-linear on every subinterval of I, and G is not regular, then the second inequality is strict.

Proof. Only the second inequality needs proof. By condition (b), and the Theorem, applied (with $n = 2$) to G, we have that

$$\frac{1}{q} \sum_{e \in E} [\phi(d(u(e))) + \phi(d(v(e)))] \geq 2\phi(\bar{d}), \tag{1}$$

and the inequality is strict if G is not regular and f has the non-linearity property. Also, by Jensen's Inequality and condition (a) we have

$$\frac{1}{p} \sum_{u \in V} \phi(d(u)) \leq \phi(\bar{d}), \tag{2}$$

and this inequality is strict if G is not regular and ϕ has the non-linearity property. We combine (1) and (2) as follows:

$$2q\phi(\bar{d}) + \sum_{e \in \tilde{E}} [\phi(d(u(e))) + \phi(d(v(e)))]$$
$$\leq \sum_{e \in E} [\phi(d(u(e))) + \phi(d(v(e)))] + \sum_{e \in \tilde{E}} [\phi(d(u(e))) + \phi(d(v(e)))]$$
$$= (p-1) \sum_{u \in V} \phi(d(u)) \leq p(p-1)\phi(\bar{d}),$$

and if G is not regular, and either f or ϕ has the non-linearity property, the resulting inequality is strict. Rearranging,

$$\frac{1}{\tilde{q}} \sum_{e \in \tilde{E}} [\phi(d(u(e))) + \phi(d(v(e)))] \leq \frac{p(p-1) - 2q}{\tilde{q}} \phi(\bar{d}) = 2\phi(\bar{d}).$$

Notice that Corollary 4 is derivable from Corollary 5 by taking $\phi(x) = x$. Of the results derivable from Corollary 5 by taking $\phi(x) = x^s$, $0 \leq s \leq 1$, Corollary 4 is the best. The silly result obtained by taking $\phi \equiv 1$ shows that the condition for strict inequality in Corollary 5 cannot be dispensed with.

We began, originally, by proving Corollaries 1 and 2 for $n = 2$ only. We are indebted to Michael Eastham for noticing the larger context (called here n-hypergraphs) to which these two results belong.

1. B. Bollobás, *Extremal Graph Theory*, Academic Press, London, 1978.
2. Hardy, Littlewood and Polya, *Inequalities*, Cambridge University Press, 1934.

Received 25 *October* 1984

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